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On the strong coupling dynamics of heterotic string theory on C^3/Z_3

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Abstract: We study the strong coupling dynamics of the heterotic $E_8 \times E_8$ string theory on the orbifolds T^6/Z_3 and C^3/Z_3 using the duality with the Horava-Witten M-theory picture. This leads us to a conjecture about the low energy description of the five dimensional E_0-theory (the CFT that describes the the singularity region of M-theory on C^3/Z_3) compactified on S^1/Z_2.

Keywords: Superstrings and Heterotic Strings, M-Theory.

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Contents

1. Introduction

Recently, models of 5D space-time bounded by two end-of the world branes attracted attention both as a laboratory for the phenomenology of elementary particle physics as well as of novel cosmological scenarios. A class of such models is inspired by compactifications of the Horava Witten (HW) M-theory [1, 2]. The understanding of the underlying 5D bulk physics is thus a key ingredient for the study of such models.

It turns out, however, that these 5D bulk theories associated with orbifold compactifications, are generally not well understood. This situation is demonstrated by the following puzzle. Consider the HW duals of the 6d T^4/Z_N heterotic orbifolds. In this case there is a localized gauge symmetry $G_1 \in E_8$ on one of the 6D end-of-the-world branes, and $G_2 \in E_8$ on the other one. Generically the heterotic spectrum includes massless twisted particles that seem to be charged under both G_1 and G_2. The puzzle is how to account for these states in the HW picture. The resolution of the puzzle follows the realization that in fact there is also a non-perturbative 7D bulk gauge theory $G_{\rm bulk}$ associated with the A_{N-1} singularity of the corresponding ALE space. The twisted states are in fact charged under say G_2 and $G_{\rm bulk}$ [3, 4] and not under G_2 and G_1. Proving the consistency of this scenario, namely, that there is a full anomaly cancellation of local symmetries, requires assigning particular boundary conditions to the 6d vector and hyper multiplets associated with the 7D vector multiplet. These boundary conditions, which seem to be quite ad hoc in the HW picture, turn out to be very natural when a duality with type I' string is invoked [5]. In the type I' picture the twisted states can be traced back to strings associated with brane junctions that involve $D6$ branes, $D8$ branes O_8 orientifold planes and $NS5$ branes.

When analyzing the HW duals of 4D heterotic orbifold models, namely, compactifications on T^6/Z_N one faces a similar puzzle. Again there are massless twisted states that are charged under both G_1 and G_2. However, since the geometry at the vicinity of the fixed points is now R^6/Z_N, which is not associated with an A_{N-1} singularity but rather with a strongly coupled E_0 theory [6], there is no room for a non-abelian non-perturbative 5D gauge symmetry. Thus, the mechanism that resolves the puzzle has to be of a different origin.

The goal of this paper is to explore the duality between the heterotic theory on the T^6/Z_3 orbifold compactification and M-theory on $(S_1/Z_2) \times (T^6/Z_3)$. In particular we would like to account for the twisted mixed states.

The compactification of M-theory on C^3/Z_3 was intensively explored [8, 6, 12, 9, 10, 11, 14]. The corresponding low energy field theory is the "mysterious" E_0 [12] theory which is a strongly coupled 5D CFT with 8 supersymmetries and a one dimensional Coulomb branch. The E_0 theory has been explored using various different techniques including non-trivial fixed points of the renormalization flow of 5D supersymmetric theories [6, 12, 11, 14], collapse of de Pezzo surfaces in Calabi-Yau compactifications [9, 10, 12, 13] and type I' string theories [13]. In spite of these study efforts and due to its strongly coupled nature, the E_0 theory is still not well understood.

Even though the full description of the E_0 theory is lacking, partial results, based on educated guesses, about the low energy description of the compactified theory can be

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obtained. This is similar to the situation with the 6D $(2,0)$ theory where after compactification on T^2, the low-energy description of the theory is given by the $N=4$ Super-Yang-Mills field theory. This field theory is interacting and is believed to correctly describe all excitations as long as their energy is much lower than the compactification scale.

In this paper we study the compactification of the E_0 theory on the segment, S^1/Z_2, with certain boundary conditions that preserve $N=1$ supersymmetry in 4D. We will propose a lagrangian that (presumably) describes the low energy excitations at a scale below the compactification scale.

The motivation for this lagrangian comes from the study of the strong coupling dynamics of the heterotic string theory on the orbifolds T^6/Z_3 and C^3/Z_3. The notation C^3/Z_3 and T^6/Z_3 is somewhat ambivalent because there are several ways to specify the action of Z_3 on the E_8 gauge degrees of freedom. In this paper we concentrate mainly on the orbifolds that break the $E_8 \times E_8$ gauge group down to $\mathrm{SU}(3) \times E_6 \times \mathrm{SU}(3) \times E_6$. We assume that the volume of T^6/Z_3 is large so that worldsheet instantons can be neglected. We analyze the moduli space of these orbifolds from the heterotic string and the low energy supergravity pictures. We discuss the local anomaly cancellation in the various scenarios.

The paper is organized as follows. The moduli space of the T^6/Z_3 orbifold is discussed in section 2 from the heterotic theory point of view. We start with a brief description of the model, its spectrum, superpotential and D- term. We then analyze the F-term flatness condition for the R^6/Z_3 case, and this condition combined with the D-term flatness for the compact case. We show that the moduli space for the non-compact case is a blow-down at the zero section of a certain line bundle over $P^2 \times P^2$. Section 3 is devoted to a brief reminder of the geometry of the blow-up of the fixed point of the C^3/Z_3 orbifold. In particular the metric, complex structure and the Euler number are written down. The moduli space is then reproduced from the supergravity description in the large blow-up limit. For completeness, we discuss gauge instantons for our model as well as the other T^6/Z_3 orbifolds. The strong coupling limit as inferred from the Horava Witten dual theory is the topic of section 6. We identify the two scales in the systems, namely, the compactification scale and the scale of the expectation value of the scalar field. We write down the $N=1$ supersymmetric 5d E_0 theory in terms of a 4D $N=1$ chiral and vector superfields. We then compactify this theory on S^1/Z_2 first in the limit of an expectation value which is much larger than the inverse of the compactification scale. In this regime we reduce the 11D HW supergravity to that of a 5D theory in the form of a non-linear sigma model. We then rewrite it in terms of a linear sigma model and determine the relations between the linear an non-linear descriptions. We then conjecture about the theory in the opposite regime where the compactification scale is larger than the inverse of the expectation value of the scalar field. Section 6 is devoted to a discussion of the anomaly cancellation in both the compact and non compact cases. In the former case the cancellation is between the contribution of the twisted states and that of the untwisted states after division by the number of fixed points. In the latter case the integration over the zero mode of the orbifold operation results in an identical division. In section 7 we summarize our results and state several open questions.

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2. The moduli space from heterotic string theory

2.1 The model

The model is the heterotic string on a T^6/Z_3 orbifold. The T^6 is of the form $T^2 \times T^2 \times T^2$ where each T^2 is a quotient of the complex plane with the root lattice of SU(3), namely, $T^2 = C/\Lambda_{\mathrm{SU}(3)}$. The tori are characterized by the identifications $z_i \sim z_i + 1$ and $z_i \sim z_i + e^{\pi i/3}$ where $i = 1, 2, 3$ and admit a Z_3 generated by the transformations

$$\Omega_1 : \ z_i \to \alpha(z_i) e^{2\pi r_i/3} z_i \,; \qquad r_i = (1, 1, -2) \,. \tag{2.1}$$

There are 27 fixed points of the α_i at $z_i = 0$, $z_i = 1/\sqrt{3}e^{\pi i/6}$, $z_i = 2/\sqrt{3}e^{\pi i/6}$.

In addition to the vector r_i the orbifolding operation is characterized also by its shift vector s_K, $k = 1, \ldots, 16$ defined by the transformation of the 16 complex left-moving fermions

$$\Omega_2 : \ \lambda^{K\pm} \to e^{\pm \frac{2\pi i s_K}{3}} \lambda^{K\pm} \,. \tag{2.2}$$

Then we define $\Omega = \Omega_1 \circ \Omega_2$ to be the generator of the Z_3 orbifold. The shift vector associated with the current model $s_K = (1, 1-2, 0^5; 1, 1, -2, 0^5)$, implies the breaking of each E_8 factor down to $E_6 \times \mathrm{SU}(3)$ (for more details see appendix A).

2.2 The spectrum, superpotential and D-term

The spectrum of the model contains the untwisted states

$$3(3, 27, 1, 1) \oplus 3(1, 1, 3, 27) \oplus 9 \text{ moduli} \tag{2.3}$$

and the twisted states

$$27(\overline{3}, 1, \overline{3}, 1) \tag{2.4}$$

where the decomposition is under $\mathrm{SU}(3) \times E_6 \times \mathrm{SU}(3) \times E_6$. (In appendix A the spectra of the other T^6/Z_3 models is presented).

Let $A = 1, \ldots, 27$ be a label of the fixed-point of Z_3 inside T_6/Z_3. Near such a fixed point the space looks like R^6/Z_3. We have one chiral field, Φ_A, in the $(\overline{3}, 1, \overline{3}, 1)$ localized around each fixed point. The moduli space is determined by the F-term and D-term conditions. The F-term comes from the superpotential:

$$W(\Phi_1, \ldots, \Phi_{27}) = \sum_A \det \Phi_A \,,$$

where we think of Φ_A as a 3×3 matrix. The D-term conditions are:

$$g^2_{\mathrm{YM}} \operatorname{tr} \sum_A \Phi^\dagger_A \tau^a \Phi_A = 0 \,, \qquad g^2_{\mathrm{YM}} \operatorname{tr} \sum_A \Phi_A \tau^a \Phi^\dagger_A = 0 \,, \qquad a = 1, \ldots, 8 \,,$$

where τ^a is a generator of SU(3) (taken as a 3×3 matrix) and g^2_{YM} is the 4D E_8 coupling constant. It is given by the 10D coupling constant divided by the volume of T^6/Z_3.

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2.3 The F-term for the orbifold R^6/Z_3

Most of the nontrivial dynamics is localized at the fixed points. So we analyze R^6/Z_3 first. Since the volume of R^6 is infinite we can set $g_{\rm YM} = 0$ and forget about the D-terms. There is only one 3×3 field Φ with superpotential $W \equiv \det\Phi$. The F-term constraints are $W' = 0$ where W' is the matrix of 2×2 minors of Φ. Thus, the F-term constraints imply that the rank of Φ is at most 1. We can therefore write Φ as:

$$\Phi = uv^T\,, \tag{2.5}$$

where u is a vector in the $(\overline{3}, 1)$ of SU(3) × SU(3) and v is a vector in the $(1, \overline{3})$. u breaks the left SU(3) down to SU(2) and v does the same to the right SU(3) so altogether we are left with SU(2) × SU(2) × U(1) where U(1) acts as:

$$u \to e^{i\theta}u\,, \qquad v \to e^{-i\theta}v\,. \tag{2.6}$$

2.4 The F-term and D-term for the orbifold T^6/Z_3

Moving back to T^6/Z_3, we solve the individual F-term constraints and find that each Φ can be written as $\Phi_A = u_A v_A^T$ $(A = 1, \dots, 27)$. The D-term constraints imply

$$0 = \sum_A (u_A^\dagger u_A)(v_A^\dagger \tau^a v_A) = \sum_A (v_A^\dagger v_A)(u_A^\dagger \tau^a u_A)\,, \qquad a = 1, \dots, 8$$

This can only be satisfied if

$$\sum_A (v_A^\dagger v_A) u_A u_A^\dagger = \sum_A (u_A^\dagger u_A) v_A v_A^\dagger = cI\,.$$

In particular, unless all of the u_A's and v_A's are zero, we need at least 3 different A's with nonzero u_A and v_A. This is because the matrix $u_A u_A^\dagger$ is of rank 1 and the sum of 2 matrices of rank 1 can never be cI (i.e. of rank 0) for $c \neq 0$.

Note that $(\Lambda_1, \Lambda_2) \in$ SU(3) × SU(3) act as

$$u_A \to \Lambda_1^\dagger u_A\,, \qquad v_A \to \Lambda_2 v_A\,.$$

If exactly 3 A's have nonzero Φ_A's (say $A = 1, 2, 3$) then we can use the SU(3) × SU(3) gauge freedom to turn the u_A's and v_A into the following form:

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_2 = v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_3 = v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this case Φ_1, Φ_2, Φ_3 are all diagonal and the unbroken symmetry is $\mathrm{U}(1)^2$. If $l > 3$ in general, all of the $\mathrm{SU}(3)^2$ will be broken.

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2.5 The moduli space for R^6/Z_3

We have seen that the moduli space for the case of R^6/Z_3 is the moduli space of all 3×3 matrices of rank ≤ 1. we can parameterize it as follows. Take $\Phi = uv^T$ as before. u and v are not uniquely defined because we can change $u\to\lambda u$ and at the same time $v\to\lambda^{-1}v$. If $\Phi\neq 0$ then neither u nor v are zero. The equivalence relation $u\sim\lambda u$ (for $\lambda\in C$) defines a point on P^2 that we will denote by $[u]$. Similarly v defines a point $[v]$ in P^2. For fixed $[u]$ and $[v]$ Φ has one more complex degree of freedom which is the overall scale and we denote it by e^σ (for $\sigma\in C$). As σ varies, e^σ spans a plane (complex line) which is fibered over $P^2\times P^2$. Let $\mathcal{L}_1$ ($\mathcal{L}_2$) be the universal line bundle over the first (second) P^2. Then,

The moduli space is the blow-down of
$\mathcal{L}_1\otimes\mathcal{L}_2$ over $P^2\times P^2$ at the zero section.

We wish to understand how this moduli space can be interpreted in the limit of small curvature and small gauge field strengths. In this case the moduli space is understood as the moduli space of the deformation of the geometry into a nonsingular space and a nonsingular instanton configuration. The space R^6/Z_3 can be deformed into a smooth space with the singularity at the origin replaced by P^2. Our goal is to argue that e^σ parameterizes the size of the P^2 (and the B-field on it) and $[u]$ and $[v]$ parameterize the $E_8\times E_8$ instanton configuration.

3. Some facts from geometry

We need some facts about the blow-up of C^3/Z_3 and the normalizable harmonic 2-form on it. In general C^n/Z_n (where the generator of Z_n acts as multiplication by $e^{2\pi i/n}$ on all the coordinates of C^n) can be deformed to a smooth (noncompact) Calabi-Yau manifold that approaches C^n/Z_n at infinity. In this CY the singular point at the origin on C^n/Z_n is replaced with a compact projective space P^{n-1}.

3.1 Metric and complex structure

We can take the metric on P^{n-1} to be the Fubini-Study metric

$$g_{i\bar{j}} = \frac{\delta_{i\bar{j}}(1+\sum|z_k|^2) - z_i\bar{z}_{\bar{j}}}{(1+\sum|z_k|^2)^2},$$

which is derived from the Kähler function

$$K = \log\left(1+|z_1|^2+\cdots+|z_{n-1}|^2\right).$$

The normalized $(1,1)$ form is $\omega = i g_{i\bar{j}} dz_i\wedge d\bar{z}_{\bar{j}}$. It can be written as $\omega = dA$ with

$$A = \frac{i}{2}e^{-K}\sum_k (z_k d\bar{z}_{\bar{k}} - \bar{z}_{\bar{k}} dz_k).$$

The blowup of C^n/Z_n can be written as a line bundle over P^{n-1} with first Chern class $c_1 = -n\omega$. Choose a constant $a>0$. Let $z_1,\ldots,z_{n-1}$ parameterize the point on the base and let $0 \le \theta < 2\pi$ be a periodic variable and $a \le r < \infty$ be a real coordinate. The metric on the blown-up space can be written as

$$ds^2 = \frac{1}{2} r^2 \sum_{i,\bar{j}=1}^{n-1} g_{i\bar{j}} dz_i d\overline{z}_{\bar{j}} + \left(1 - \frac{a^{2n}}{r^{2n}}\right)^{-1} dr^2 + \frac{r^2}{n^2}\left(1 - \frac{a^{2n}}{r^{2n}}\right)(d\theta - nA)^2 \,.$$

The holomorphic coordinates are $z_1, \ldots, z_{n-1}$ and

$$w = \sqrt{r^{2n} - a^{2n}} e^{\frac{K}{2} + i\theta} \,.$$

This space has a normalizable 2-form:

$$\widetilde{F} = \frac{2(n-1)a^{2(n-1)}}{nr^{2n-3}} dr \wedge (d\theta - nA) + \frac{a^{2(n-1)}}{r^{2(n-1)}} \omega \,. \tag{3.1}$$

It satisfies

$$\int_{P^1 \subset P^{n-1}} \widetilde{F} = 2\pi \,, \qquad \int_{\text{Fiber}} \widetilde{F} = \frac{2\pi}{n} \,. \tag{3.2}$$

Also

$$\int \widetilde{F} \wedge \widetilde{F} \wedge \cdots \wedge \widetilde{F} = \frac{2\pi}{n^2} \int \omega^{n-1} = \frac{(2\pi)^n}{n^2} \,. \tag{3.3}$$

Here P^{n-1} denotes the divisor given by $r = a$ and $P^1 \subset P^{n-1}$ is any (complex) line inside this divisor. The fibers are given by fixed z_k's. For $n = 3$ we can calculate

$$-\frac{1}{2} \int_{P^2} \mathrm{tr}_6 \, R \wedge R = 12(2\pi)^2 \,.$$

The trace is in the representation $\mathbf{6}$ of SO(6).

Another property of interest is the Euler number of the T^6/Z_n orbifolds [15]. Consider in particular the $n = 3$ case. At each fixed point one deletes the singular point and glues instead, as was explained above, a non-compact manifold composed of a P^2 and a line bundle over it with a resulting $c_1 = 0$. The Euler number of each of these glued manifolds is $\chi = 3$. Since T^2 has a vanishing χ, the torus with the 27 deleted points has $\chi = -27$. Thus, the Euler number of T^6/Z_3 is $\chi = -27/3 = -9$. We add now the Euler number of the glued manifolds and we end with $\chi = -9 + 27 \times 3 = 72$.

4. The moduli space from supergravity

We would like to reproduce the moduli space found in section 2.5 in the limit that the blow-up parameter, a, is large compared to l_s. The moduli space is then the moduli space of $E_8 \times E_8$ instantons on the blown-up C^3/Z_3. The requirement on the instanton is that the holonomy of the gauge field at infinity is known. Namely, define the contour

$$\gamma(t) = (z_1 = \xi_1, \ldots, z_{n-1} = \xi_{n-1}, \theta = t, r = r_0) \,, \qquad 0 \le t < 2\pi \,.$$

where $\xi_1, \ldots, \xi_{n-1}, r_0$ are constants. We require that as $r_0 \to \infty$ the holonomy of the $E_8 \times E_8$ gauge field along $\gamma(t)$ is conjugate to Ω_2 that was defined in (2.2). The second requirement is that

$$\frac{1}{30} \int_{P^2} \sum \mathrm{tr}_{248} \, F \wedge F = \int_{P^2} \mathrm{tr} \, R \wedge R = -24(2\pi)^2 \,.$$

Here P^2 is the exceptional divisor of the blow-up. The instanton solutions that we will consider are abelian. They are given by picking a generator τ of the Lie algebra of $E_8 \times E_8$, normalized such that $e^{2\pi i\tau}$ is the unity in the group, and setting the $E_8 \times E_8$ field-strength to be $F = i\widetilde{F}\tau$, where $\widetilde{F}$ is the 2-form found in (3.1). (Note that F is defined to be anti-hermitean.) According to (3.2), the holonomy around $\gamma(t)$ is $e^{\frac{2\pi i}{3}\tau}$. Thus τ should be chosen so that $e^{\frac{2\pi i}{3}\tau}$ will be Ω_2 — the generator of Z_3 in the gauge group. The instanton condition implies that

$$24 = \frac{1}{30}\sum \operatorname{tr}_{248}\tau^2 .$$

For the Ω_2 that breaks $E_8 \times E_8$ to $(E_6 \times \mathrm{SU}(3))^2$ we can pick τ as follows. Let $\phi_1 : \mathrm{SU}(3) \to E_8$ be the embedding of SU(3) in the first E_8 factor and let ϕ_2 be the embedding in the second factor. We pick $\widetilde{\tau} \in \mathrm{SU}(3)$ to be $\operatorname{diag}(1, 1, -2)$ and take $\tau = \phi_1(\widetilde{\tau}) \oplus \phi_2(\widetilde{\tau})$. Note that for elements of E_8 that are embedded in SU(3) we have

$$\frac{1}{30}\operatorname{tr}_{248}\phi_i(\widetilde{\tau})^2 = 2\operatorname{tr}_3\widetilde{\tau}^2 = 12 .$$

Summing the contributions of the two E_8 factors gives 24. The moduli space is the moduli space of embeddings of τ inside $E_8 \times E_8$ keeping the holonomy at infinity, $e^{\frac{2\pi i}{3}\tau}$ fixed. The moduli space of different $\widetilde{\tau}$'s in SU(3) that are conjugate to $\operatorname{diag}(1, 1, -2)$ is P^2. Thus the moduli space of instantons is $P^2 \times P^2$. The coordinate σ from (2.5) is interpreted as $\sigma = \frac{1}{2}\alpha'\pi a^2 + iB$ where πa^2 is the area of a P^1 divisor inside the exceptional P^2 and B is the integral of the NSNS 2-form on P^1.

It is interesting to check other embeddings of Z_3 inside $E_8 \times E_8$. In one embedding we take the generator of Z_3 to be in the center of $\mathrm{SU}(9)/Z_3 \subset E_8$. The element is $e^{2\pi i/9}$. We then take $\widetilde{\tau} \in \mathrm{su}(9)$ to be

$$\widetilde{\tau} = \operatorname{diag}\left(\underbrace{\frac{1}{3}, \ldots, \frac{1}{3}}_{8}, -\frac{8}{3}\right).$$

We let $\phi : \mathrm{su}(9) \to E_8$ be the embedding and we calculate

$$\frac{1}{30}\operatorname{tr}_{248}\phi(\widetilde{\tau})^2 = 2\operatorname{tr}_9\widetilde{\tau}^2 = 16 .$$

Therefore, in the second E_8 factor we should find an embedding with instanton number 8. For this we break E_8 to $\mathrm{SU}(14) \times \mathrm{U}(1)$ and embed τ as $e^{2\pi i/3}$ inside U(1).

5. Strong coupling limit

At strong coupling, the $E_8 \times E_8$ heterotic string theory on C^3/Z_3 is described by weakly coupled 11D gravity on the bulk of $(S^1/Z_2) \times (C^3/Z_3) \times R^{4,1}$ and $G \in E_8$ gauge fields on the two boundaries. The only strongly coupled part of this background comes from the fixed point of the Z_3 action. It is described by the E_0 theory compactified on S^1/Z_2.

The E_0 theory is a strongly coupled 5D CFT [12] that describes the localized degrees of freedom of M-theory on C^3/Z_3 at the fixed point. We will recall some known facts about the E_0-theory in section 5.2. We will then use the heterotic string analysis of the orbifold to make conjectures about the 4D low-energy description of the E_0 theory compactified on S^1/Z_2.

5.1 The setting

The E_0-theory is compactified on an interval S^1/Z_2 and, as we shall see, there are extra degrees of freedom on the boundaries. Let the length of the interval be πR. The 5D E_0-theory has a moduli space R/Z_2 [6] that is parameterized by an order parameter χ_0 with dimensions of mass (see section 5.2 below).

Our problem has two scales:

- The *compactification scale*, $1/R$,
- The E_0-*scale*, $\chi_0 \equiv \langle\chi\rangle$.

If $\chi_0 \gg 1/R$ the 5D compactification (energy) scale is low and we can first reduce the E_0-theory to its 5D low-energy description and then compactify the latter on S^1/Z_2 with appropriate boundary conditions.

If, on the other hand, the condition $\chi_0 \gg 1/R$ is not met, quantum corrections are strong and we do not know the metric on the moduli space of the effective 4D low-energy theory. Since we do not know any microscopic definition for χ_0 it does not make sense to use χ_0 as a coordinate on the 4D moduli space. We can still look for an effective 4D low-energy description but it requires a better understanding of the E_0 theory. We will propose a conjecture about that description in section 5.8.

5.2 The E_0-theory

The E_0-theory is a five-dimensional interacting CFT. It has $N = 1$ supersymmetry (8 generators) in 5D. It has a one-dimensional Coulomb branch parameterized by a real coordinate $\chi_0 > 0$. For a generic χ_0 the low-energy description is a 5D vector multiplet with (five dimensional) $N = 1$ supersymmetry. This multiplet comprises of a scalar χ (whose VEV is χ_0), a vector field A and fermions. The low-energy effective action is

$$L = \frac{1}{4\pi^2}\int\left[\frac{1}{2}\chi\partial_\mu\chi\partial^\mu\chi + \frac{1}{4}\chi F_{\mu\nu}F^{\mu\nu} + \frac{1}{24}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}A_{\mu_1}F_{\mu_1\mu_2}F_{\mu_3\mu_4}\right]d^5x + (\text{fermions}). \tag{5.1}$$

Here χ is a scalar field of dimension 1, A is a low-energy U(1) gauge field and $F = dA$. The Coulomb branch is parameterized by the VEV $\chi_0 = \langle\chi\rangle$. The E_0-theory has a rich structure of electric BPS particles with masses that are integer multiples of χ_0 and magnetic BPS strings with tensions that are integer multiples of χ_0^2 [8, 9, 6].

We can rewrite (5.1) using 4D $N = 1$ superspace. We separate one direction out of the five, call it x_4, and we define the chiral superfield $\widetilde{\Phi}(x_4)$ and the vector superfield $\widetilde{V}(x_4)$ (which depend explicitly on the parameter x_4) so that

$$\widetilde{\Phi}(x_4)|_{\theta=\overline{\theta}=0} = \chi(x_4) + iA_4(x_4) \tag{5.2}$$

and, in the WZ gauge,

$$\widetilde{V} = -\theta\sigma^\mu\overline{\theta}A_\mu + i\theta^2\overline{\theta}\lambda - i\overline{\theta}^2\theta\overline{\lambda} + \frac{1}{2}\theta^2\overline{\theta}^2 D \,. \tag{5.3}$$

The gauge freedom is

$$\widetilde{V} \to \widetilde{V} + \Lambda + \Lambda^\dagger \,, \qquad \widetilde{\Phi} \to \widetilde{\Phi} + 2\partial_4\Lambda \,.$$

where Λ is a superfield. Without any boundary conditions the lagrangian is

$$\begin{aligned} L = &-\frac{1}{12\pi^2}\int\left[\frac{1}{8}(\widetilde{\Phi}+\widetilde{\Phi}^\dagger-2\partial_4\widetilde{V})^3+\widetilde{V}\partial_4\widetilde{V}D_\alpha W^\alpha+\partial_4\widetilde{V}\overline{D}^{\dot\alpha}\widetilde{V}\overline{W}_{\dot\alpha}+\partial_4\widetilde{V}D_\alpha\widetilde{V}W^\alpha\right]d^4\theta - \\ &-\frac{1}{16\pi^2}\int\widetilde{\Phi}W_\alpha W^\alpha d^2\theta-\frac{1}{16\pi^2}\int\widetilde{\Phi}^\dagger\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}d^2\overline{\theta} \end{aligned}$$

where, as usual, $W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha\widetilde{V}$ and it satisfies $D^\alpha W_\alpha = \overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha}$.

It is not hard to check that this lagrangian is gauge invariant up to a total derivative

$$\delta\int Ld^4x = -\frac{1}{24\pi^2}\partial_4\int\Lambda W_\alpha W^\alpha d^2\theta d^4x - \frac{1}{24\pi^2}\partial_4\int\Lambda^\dagger\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}d^2\overline{\theta}d^4x$$

Expanded in components we find

$$\begin{aligned} L = &\frac{1}{4\pi^2}\int\left[\frac{1}{2}\chi(\partial_\mu\chi)^2+\frac{1}{2}\chi F_{4\mu}F^{4\mu}+\frac{1}{2}\chi^2\partial_4 D-\frac{1}{2}\chi F^*F\right]d^5x + \\ &+\frac{1}{4\pi^2}\int\left[-\frac{1}{2}\chi D^2+\frac{1}{4}\chi F_{\mu\nu}F^{\mu\nu}+\frac{1}{8}A_4\epsilon^{\mu\nu\rho\tau}F_{\mu\nu}F_{\rho\tau}-\frac{1}{6}A_\mu\epsilon^{\mu\nu\rho\tau}\partial_4 A_\nu F_{\rho\tau}\right]d^5x \end{aligned} \tag{5.4}$$

Here $\mu = 0,\ldots,3$. Although this action does not look manifestly Lorentz invariant, if we eliminate the auxiliary fields we find $D = -\partial_4\chi$ and $F = 0$ and the whole action can be rewritten as (5.1) and the action is manifestly Lorentz invariant (although supersymmetry is not manifest).

So far we have discussed the low-energy effective action in 5D. Later on we will discuss a compactification on S^1/Z_2 that breaks half the supersymmetry. For completeness, we note that the simplest compactification of the E_0 down to 4D is on S^1 in such a way that preserves $N = 2$ supersymmetry. We will take the coordinate along S^1 to be $0 \le x_4 \le 2\pi R$. The low-energy description of the E_0-theory compactified on S^1 is given by a single $N = 2$ U(1) vector multiplet with the effective action that is derived from the Seiberg-Witten curve:

$$y^2 = (x^2 - u)(x - u^2)\,.$$

Here u is a holomorphic parameter on the moduli space. For $|u| \to \infty$ we can write

$$\frac{1}{2\pi R}\log u \approx \langle\chi + iA_4\rangle\,.$$

5.3 Adding a boundary

We take the effective action (5.1) and compactify it on the interval S^1/Z_2. Let $0 \leq x_4 \leq \pi R$ be the coordinate along the interval (the other coordinates are $x_0, \dots, x_3$). Now we add the boundaries at $x_4 = 0$ and $x_4 = \pi R$. Our goal is to propose a lagrangian that describes the theory at energies

$$E \ll \frac{1}{R} \ll \chi_0 \equiv \langle \chi \rangle \,.$$

In the context of M-theory, as we explained above, the compactification arises from M-theory on a blown up C^3/Z_3 where the singularity was blown up to a P^2 with volume $M_p^{-6}\chi_0^2$. We are interested in the regime $\chi_0 \ll M_p$ where the scale of our low-energy E_0-theory is lower than M_p.

The U(1) gauge group of the low-energy effective action (5.1) has an anomaly when we add a boundary. This anomaly comes from the Chern-Simons term

$$\frac{1}{32\pi^2}\int A\wedge F\wedge F \equiv \frac{1}{32\pi^2}\int \epsilon^{\alpha\beta\gamma\mu\nu} A_\alpha F_{\beta\gamma} F_{\mu\nu} d^5x \,. \tag{5.5}$$

This term comes from the Chern-Simons term of 11D supergravity reduced on the blown-up C^3/Z_3 with the substitution $C = 3A(x_0, \dots x_4)\wedge\widetilde{F}(x_5, \dots, x_{10})$. Here $\widetilde{F}$ is the harmonic 2-form defined in (3.1). The factor of 3 is needed because it is $3\widetilde{F}$ that has integral periods according to (3.3). In 5D, the Chern-Simons term must be an integer product of $\frac{1}{96\pi^2}\int A\wedge F\wedge F$ (see [8]). Our model has 3 times the fundamental unit of anomaly.

There are two possible ways to eliminate the anomaly. We can either add degrees of freedom on the boundary with the opposite anomaly or we can set the boundary conditions on the gauge field so that its parallel components vanish on the boundary. In that case, the gauge group contains only transformations that vanish on the boundary and there is no anomaly.

In the next subsections we will discuss two descriptions of the theory with a boundary. We will start with an analysis of the regime $1/R \ll M_p \ll \chi_0$ where the Horava-Witten supergravity approximation to M-theory is valid. We will see that the dimensional reduction of the Horava-Witten low-energy effective action leads to a description where the gauge transformation vanish on the boundary. Extra degrees of freedom on the boundary are then described by a nonlinear σ-model.

We will then discuss an alternative description in terms of a linear σ-model where the gauge transformations are allowed not to vanish on the boundary. Instead extra charged chiral superfields cancel the anomaly on the boundary.

5.4 Description in terms of the nonlinear σ-model

When $M_p \ll \chi_0$ the volume of the P^2 blow-up is large and we can reduce the Horava-Witten supergravity lagrangian — the 11D supergravity in the bulk and E_8 gauge fields on the 10D boundary — down to the noncompact 5D. The reduction of the E_8 gauge fields was discussed in detail in section 4 and the result was that the low-energy degrees of freedom describe the moduli space of a certain instanton. The moduli space was described as the embedding of $\mathrm{U}(1) \in \mathrm{SU}(3)$ where SU(3) was a fixed subgroup of E_8. The E_8 instanton moduli space in this problem is therefore a copy of P^2 (not to be confused with the blow up divisor which is also a P^2 but in space-time and not in field space). Thus, at low-energies, the degrees of freedom on the boundary are described by a nonlinear σ-model with P^2 as the target space. The appropriate boundary conditions for the superfields $\widetilde{\Phi}$ and $\widetilde{V}$, defined in (5.2)–(5.3), can be obtained from the Horava-Witten boundary conditions on the 3-form field C of 11D supergravity. Recall that Horava and Witten described the segment $[0, \pi R]$ as S^1/Z_2 with the Z_2 acting as

$$\begin{aligned} Z_2 :\ & C_{10\mu\nu}(x_{10}) \to C_{10\mu\nu}(-x_{10})\,, \\ Z_2 :\ & C_{\sigma\mu\nu}(x_{10}) \to -C_{\sigma\mu\nu}(-x_{10})\,, \end{aligned}$$

Here $\mu, \nu = 0, \ldots, 9$. In our problem x_{10} should be replaced with x_4. The connection between C and the components A_μ and A_4 of $\widetilde{V}$ and $\widetilde{\Phi}$ is

$$C_{\mu IJ} = A_\mu \widetilde{F}_{IJ}\,, \qquad C_{4IJ} = A_4 \widetilde{F}_{IJ}\,.$$

where $\widetilde{F}$ is the 2-form on the blown up C^3/Z_3 that was defined in (3.1). It follows that Z_2 acts on A_μ and A_4 as

$$A_\mu(x_4) \to -A_\mu(-x_4)\,, \qquad A_4(x_4) \to A_4(-x_4)\,.$$

These rules can be extended to the superfields and we can now derive the boundary conditions

$$\widetilde{V}|_{x_4=0} = \widetilde{V}|_{x_4=\pi R} = 0\,, \qquad \partial_4\widetilde{\Phi}|_{x_4=0} = \partial_4\widetilde{\Phi}|_{x_4=\pi R} = 0\,.$$

5.5 The linear σ-model

We will now discuss an alternative description of the boundary. The Chern-Simons term (5.5) produces an anomaly for the U(1) gauge transformations on the boundary. This anomaly can be canceled by 3 4D chiral fermions with U(1) charge 1 on the boundary. Thus, we assume that on the boundary there is a chiral superfield $\widetilde{X}$ that is in the fundamental representation 3 of a global SU(3) and is also charged under the 5D U(1) gauge field. The total action now has an extra term

$$\int \widetilde{X}^\dagger e^{\widetilde{V}} \widetilde{X} d^4\theta d^4x\,. \tag{5.6}$$

On the other hand we do not impose Dirichlet boundary conditions on the gauge fields. When we integrate out the auxiliary field D from (5.4) we need to integrate by parts over dx_4. When there is a boundary we find a boundary contribution of

$$\frac{1}{2}\chi^2 D\,.$$

The field D also appears linearly in (5.6). Integrating D out we find the boundary condition

$$\frac{1}{2}\chi^2|_{x_4=0} = |X|^2\,, \qquad X \equiv \widetilde{X}|_{\theta=\overline{\theta}=0}\,, \tag{5.7}$$

where X is the scalar ($\theta = \overline{\theta} = 0$) component of $\widetilde{X}$. Note also that the a single U(1) charged chiral superfield is anomalous in 4D but the anomaly is canceled by the inflow from the term $\int A\wedge F\wedge F$ in (5.4) in 5D. This consideration also shows that we need exactly 3 chiral fields to cancel the anomaly from the 5D WZ term.

In the regime $\chi_0 \gg \frac{1}{R}$ the other boundary is far and the effective action near the boundary cannot have any dimensionful parameter in it. The superfields $\widetilde{\Phi}$ and $\widetilde{X}$ both have mass dimensions of 1. This implies that there is a possibility to add

$$I_2 = C_0 \int_{x_4=0} (\widetilde{\Phi} + \widetilde{\Phi}^\dagger - 2\partial_4 \widetilde{V})^2 d^4\theta\, d^4x - \frac{C_1}{4} \int_{x_4=0} \left[\int W_\alpha W^\alpha d^2\theta + \int \overline{W}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}} d^2\overline{\theta} \right].$$

to the action. Here C_0 and C_1 are unknown coefficient. Such a term will not affect the boundary conditions (5.7).

Similarly, we add a chiral superfield $\widetilde{Y}$ at the other end, $x_4 = \pi R$. The U(1) anomaly cancellation can be satisfied if we require that $\widetilde{Y}$ is also a triplet and has charge -1 under the 5D U(1) gauge group. We assume that it is in the representation 3 of another global SU(3). $\widetilde{Y}$ satisfies the boundary condition

$$\frac{1}{2}\chi^2|_{x_4=\pi R} = |Y|^2 , \qquad Y \equiv \widetilde{Y}|_{\theta=\overline{\theta}=0} .$$

5.6 Relation between the linear and nonlinear σ-models

We can now solve the equations of motion of the nonlinear σ-model at energies $E \ll 1/R$. If we vary the action with respect to A_4 and integrate by parts, keeping boundary terms, so as to extract the coefficient of $\delta A_4(0)$ we find the equation

$$0 = \chi^2 A_\mu + \mathrm{Im}(X_i^* D_\mu X_i) + \chi F_{4\mu} + C_0 \partial^4 F_{\mu 4} + C_1 \partial^\nu F_{\mu\nu} .$$

The first term comes from $|D_\mu X|^2$ and the boundary condition (5.7). To look for the zero modes we assume that X_i, A_4 and A_μ are independent of $x_0, \ldots, x_3$. We find the boundary condition at $x_4 = 0$:

$$0 = \chi_0^2 A_\mu + \chi_0 \partial_4 A_\mu - C_0 \partial^4 \partial_4 A_\mu .$$

There is a similar equation at $x_4 = \pi R$. The equation of motion in the bulk is $\partial_4 \partial^4 A_\mu = 0$ and for $\chi_0 \neq 0$ there is no zero mode because any nonzero solution that is independent of $x_0, \ldots, x_3$ cannot satisfy both boundary conditions.

Now we allow the fields to vary as a function of $x_0, \ldots, x_3$. For low-energy modes, we can neglect $\partial^\nu F_{\nu\mu}$ and $\partial^4 \partial_4 A_\mu$ and $\chi_0 \partial^4 A_\mu$ compared to $\chi_0^2 A_\mu$. We therefore get

$$A_\mu|_{x_4=0} \approx -\frac{1}{\chi_0^2} \mathrm{Im}(X_i^* D_\mu X_i) + \frac{1}{\chi_0} \partial_\mu A_4$$

and a similar equation at the other end. Together we find the solution:

$$A_\mu(x_4) \approx -\frac{\pi R - x_4}{\pi R \chi_0^2} \mathrm{Im}(X_i^* D_\mu X_i) - \frac{x_4}{\pi R \chi_0^2} \mathrm{Im}(Y_i^* D_\mu Y_i) + \frac{1}{\chi_0} \partial_\mu A_4 .$$

Assuming that $R\chi_0 \gg 1$ our assumptions about neglecting $\chi_0 \partial_4 A_\mu$ are correct. We also assume that A_4 is independent of x_4.

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5.7 The moduli space

The low energy effective action in 4D is a nonlinear σ-model with a 5 (complex) dimensional target space. The target space can be described as a line bundle over $P^2 \times P^2$, the same as the one discussed in section 2.5. The homogeneous coordinates on the first P^2 are given by the fields $X_i/\|X\|$ $(i = 1, \ldots, 3)$ on the boundary at $x_4 = 0$ and the homogeneous coordinates on the second P^2 are given by $Y_i/\|Y\|$. The coordinates on the fiber of the line bundle is the $\theta = \overline{\theta} = 0$ component of the superfield $e^{\int_0^{\pi R} \widetilde{\Phi}(x_4)dx_4}$. We will denote it by z. The symmetry group $\mathrm{SU}(3) \times \mathrm{SU}(3)$ acts non-trivially on the moduli space and the orbit of any point is an S^1 bundle over $P^2 \times P^2$.

Finally, let us discuss the metric on the moduli space. For $R\chi_0 \gg 1$, the size of each P^2 is given by

$$\|X\| = \|Y\| = \frac{1}{\sqrt{2}}\chi_0 = \frac{1}{\sqrt{2}\pi R}\log|z|\,.$$

The metric on the fiber is

$$\frac{\chi_0}{\pi R}\frac{|dz|^2}{|z|^2} = \frac{\log|z|}{\pi^2 R^2 |z|^2}|dz|^2\,.$$

5.8 The regime opposite to $\chi_0 \gg 1/R$

If the condition $\chi_0 \gg 1/R$ is not met then χ_0, being defined as the 5D VEV, is not well defined. We cannot use the low-energy effective action (5.1) but we will propose below an alternative lagrangian that describes the compactified theory at energies $E \ll 1/R$. There are several examples of 5D and 6D strongly interacting theories that after compactification to 4D are described at low energies by ordinary field theories. The $(2,0)$ theory, for example, is described by $N = 4$ Super-Yang-Mills theory after compactification on T^2 and at energies below the compactification scale. We can hope that the E_0 theory compactified on the segment S^1/Z_2 (with the boundary conditions implied by the M-theory construction above) is also described by a regular field theory at energies below the compactification scale. The heterotic string analysis that was discussed in section 2.2 provides a clue.

Before tackling the entire spectrum up to scales $E \ll 1/R$ let us consider the lowest end of the spectrum, namely the moduli space. The starting point is the classical result of section 5.7. $N = 1$ supersymmetry allows for quantum corrections to both the Kähler metric as well as the superpotential. However, we assume that the global $\mathrm{SU}(3) \times \mathrm{SU}(3)$ remains a good symmetry (as the embedding into string theory suggests). Since the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ orbit of a point in the classical 5(complex) dimensional moduli space found in section 5.7 is of real codimension 1 there cannot be any generated superpotential (since there is no nonzero holomorphic function that is constant on a real codimension 1 subspace and zero at infinity). The Kähler potential, on the other hand, can receive quantum corrections. The $\mathrm{SU}(3) \times \mathrm{SU}(3)$ symmetry imposes restrictions on the possible Kähler metrics. To see what the restrictions are it is convenient to parameterize the moduli space by a complex 3×3 matrix, Φ_1, of rank 1. When the variables $\widetilde{X}, \widetilde{Y}$ and $\widetilde{\Phi}$ discussed in section 5.5 are valid, Φ_1 can be taken as the $\theta = \overline{\theta} = 0$ component of the superfield

$$R\widetilde{X}e^{\int_0^{\pi R} \widetilde{\Phi}(x_4)dx_4}\widetilde{Y}^T\,. \tag{5.8}$$

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The only $\mathrm{SU}(3) \times \mathrm{SU}(3)$ invariant that can be constructed from a rank 1 matrix is $\xi \equiv \operatorname{tr} \Phi_1^\dagger \Phi_1$. Note that in the regime $\chi_0 \gg 1/R$ we have

$$\xi \equiv \operatorname{tr} \Phi_1^\dagger \Phi_1 = R^4 \chi_0^4 e^{2\pi R \chi_0} \,.$$

The $\mathrm{SU}(3) \times \mathrm{SU}(3)$ invariant Kähler metric can only be a function of the real variable ξ. In the regime ($\xi \gg 1$) the Kähler function is proportional to χ_0^3 as can be seen from (5.1) (and is a complicated function of ξ). $K(\xi)$ receives quantum corrections and we wish to extrapolate to the region of small ξ. We will assume that there is a point $\xi = 0$ in the moduli space where the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ symmetry is restored. In principle, the point $\xi = 0$ might be infinitely far away on the moduli space but this seems unlikely.

If such a point where $\mathrm{SU}(3) \times \mathrm{SU}(3)$ is restored exists we might hope that it is described at low energies by an ordinary field theory. We conjecture that this is indeed the case and that for $\xi \ll 1$ the dynamics at energies $E \sim \xi^{1/2}/R \ll 1/R$ is reproduced by the following lagrangian. First, we should elevate the field Φ_1 defined in (5.8) to a generic 3×3 matrix, Φ, without any restriction on the rank. The lagrangian is then

$$\int \operatorname{tr} \Phi^\dagger \Phi d^4\theta + \lambda \int \det \Phi d^2\theta + \lambda \int \det \Phi^\dagger d^2\overline{\theta} \,. \tag{5.9}$$

Here λ is an unknown real constant.

We have seen in section 2.3 that the potential of the scalar component of Φ in (5.9) has a minimum when the 3×3 matrix has rank 1 (or 0). The moduli space of 3×3 matrices of rank 1 can be parameterized as in (2.5) and using (5.8) we see that the massless spectrum is reproduced correctly. In other words, at energies $E \ll \|\sqrt{\langle \operatorname{tr} \Phi^\dagger \Phi \rangle}\|$ the dynamics that is described by (5.9) and the dynamics that is described in (5.5) coincide.

The global symmetry $\mathrm{SU}(3) \times \mathrm{SU}(3)$ acts on Φ as $\Phi \to \Lambda_1 \Phi \Lambda_2$ with $(\Lambda_1, \Lambda_2) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$. Up to an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ transformation we can choose the VEV $\langle \Phi \rangle$ to be of the form

$$\langle \Phi \rangle = \begin{pmatrix} \phi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

It breaks $\mathrm{SU}(3) \times \mathrm{SU}(3)$ down to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, as discussed in (2.3). The 9 complex fields that comprise Φ decompose under $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as

$$(1,1)_0 + (2,2)_0 + (1,2)_3 + (2,1)_{-3} \,.$$

Here the subindex denotes the $\mathrm{U}(1)$ charge. The 4 fields in the representations $(2,2)_0$ are massive with masses of order ϕ_0. Note that if a particle with the quantum numbers $(2,2)_0$ existed it could decay into the massless particles with quantum numbers $(1,2)_3$ and $(2,1)_{-3}$. Moreover, we expect λ to be a parameter of order 1 and therefore (5.9) is strongly coupled.

The conjecture (5.9) is motivated by the string theory effective action. The string theory derivation assumes, of course, that $\lambda_s^{1/3} = M_p R \ll 1$ but we conjecture that the form above remains valid even when $M_p R \gg 1$. The reason is that we expect the 4D low energy effective action to be scale invariant at energies well below $1/R$. The F-term in the expression (5.9) receives no quantum corrections and therefore the dimension of Φ can receive no quantum corrections. The Kähler term can receive quantum correction. By dimensional analysis it should be of the form $\Phi^\dagger\Phi f(R\Phi, R\Phi^\dagger)$ where f is some function of the dimensionless quantities $R\Phi$ and $R\Phi^\dagger$. Assuming that $f(0,0) \neq 0$ we can normalize Φ such that $f(0,0) = 1$ and for small $R\Phi$ we can approximate f to be a constant.

Note that Φ has 9 components but when Φ gets a nonzero VEV that is a matrix of rank 1 only 5 components of Φ remain massless (since a rank 1 matrix can be put in the form (2.5) with the equivalence relations (2.6)). Therefore, 4 components of Φ are massive at a generic point of the moduli space. The masses of these components are of the order of χ_0 and when $\chi_0 \sim 1/R$ the masses are of the order of the compactification scale. Since we are neglecting any modes with masses of the order of $1/R$, the form (5.9) does not contain any information in addition to the moduli space for $\chi_0 \sim 1/R$.

6. Anomalies

In this section we discuss the cancellation of anomalies of local symmetries in the 4D field theories associated with the Z_3 orbifold models. We consider here the case of $[E_6 \times \mathrm{SU}(3)] \times [E_6 \times \mathrm{SU}(3)]$ gauge symmetry, and the rest of the Z_3 models in the appendix A. We analyze the anomalies of the low energy effective action of the heterotic string theory compactified on compact T^6/Z_3, on non-compact C^3/Z_3, for the HW dual model and for the case where the singularity is blown up.

6.1 The T^6/Z_3 orbifold model

At each fixed point anomalies can potentially occur only in the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ subgroup of the local symmetry group. Recall that the twisted states are in the representation $(\bar{3}, \bar{3})$ so the contribution to the anomaly of each SU(3) is that of 3 anti-fundamental representations. The charged untwisted massless matter transform in the $3(3, 27, 1, 1) \oplus 3(1, 1, 3, 27)$ representation of the full symmetry group. Thus, the contribution to the anomaly of each SU(3) symmetry group is that of 81 fundamentals divided evenly between all the fixed points, namely divided by 27 — the number of fixed points. So altogether the contribution of the untwisted matter is that of three fundamentals. Hence there is an exact cancellation between the twisted and untwisted states and both SU(3) gauge symmetries are anomaly free.

6.2 The C^3/Z_3 orbifold model

Consider now the heterotic compactification on non-compact R^6/Z_N Calabi Yau orbifolds. In such a case, it may seem that one faces a problem with the cancellation of local anomalies. Again we consider the Z_3 model with the unbroken gauge group $[E_6 \times \mathrm{SU}(3)] \times [E_6 \times \mathrm{SU}(3)]$.

The content of the massless spectrum is the same as that of the compact case. The main difference is that unlike the 27 fixed points of the compact case here there is only one single singular point, namely, the origin. Thus it seems that there is not reason to divide the contribution of the untwisted sector by 27 and hence it looks as if the anomaly associated with the twisted matter cannot balance that of the untwisted sector.

It turns out that the division that one invokes in the compact case due to the multiplicity of fixed points, should be implemented also in the non-compact case. As was shown by Gimon and Johnson [16], when performing the corresponding one loop stringy computation in the non-compact case, there is a zero mode of the Z_3 projection that one has to take into account. The trace of the twist operator $\alpha = e^{\frac{2\pi ik}{N}}$ that operates on the complex coordinates $z \to e^{\frac{2\pi ik}{N}} z$ for each T^2 takes the form

$$\mathrm{Tr}[e^{\frac{2\pi ik}{N}}] = \int dz d\bar{z} \langle z, \bar{z}|\alpha|z, \bar{z}\rangle = \frac{1}{4\sin^2 \frac{\pi k}{N}} ,$$

where one uses the basis with $\langle z|z'\rangle = \frac{1}{V_{T^2}}\delta(z - z')$. For the T^6 and for a Z_3 orbifold we thus get the factor of $1/(4\sin^2 \frac{\pi k}{3})^3 = 1/27$ which we have to multiply the contribution of the untwisted sector. Therefore, like for the compact case, there is an exact cancellation between the contributions of the twisted and untwisted sectors to the anomaly.

6.3 The HW dual with blown up fixed point

In the HW dual of both the compact and non compact orbifold models one has to cancel the 4D anomalies locally at each point along the S^1/Z_2 interval and in particular at the two ends of it.

We will discuss now the anomaly cancellation for the blown up R^6/Z_3 case and at the end of this subsection we comment on the case where the fixed points are not blown up.

Let us first see how the anomalies are canceled when we are at a generic point of the moduli space where the symmetry is broken down to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$. Let us also denote the 5d U(1) gauge group as $\mathrm{U}(1)_B$.

The fields $\widetilde{X}$ have the following $(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R)_{(\mathrm{U}(1)_V, \mathrm{U}(1)_B)}$ quantum numbers:

$$\widetilde{X} : (2,1)_{(1,1)} + (1,1)_{(-2,1)} , \qquad \widetilde{Y} : (1,2)_{(-1,-1)} + (1,1)_{(2,-1)} ,$$

We can take the contribution of the untwisted fields to the anomaly to as that of 3 superfields each with quantum numbers

$$(2,1)_{(-1,0)} + (1,1)_{(2,0)} + (1,2)_{(1,0)} + (1,1)_{(-2,0)} .$$

When $\mathrm{SU}(3) \times \mathrm{SU}(3)$ is broken down to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ the VEVs of $\widetilde{X}$ and $\widetilde{Y}$ are not invariant under $\mathrm{U}(1)_V$. We can rectify this with a compensating $\mathrm{U}(1)_B$ gauge transformation. In other words, we define a $\mathrm{U}(1)_C$ such that the charges satisfy $Q_C \equiv Q_V + 2Q_B$. The $(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R)_{\mathrm{U}(1)_C}$ quantum numbers are now:

$$\widetilde{X} : (2,1)_3 + (1,1)_0 , \qquad \widetilde{Y} : (1,2)_{-3} + (1,1)_0 .$$

We can now take the VEVs to be invariant under $\mathrm{U}(1)_C$. We can now check the local cancellation of the anomaly. The fields $\widetilde{X}$ together with the untwisted fields on the $x_4 = 0$ end with quantum numbers $(2,1)_{-1} + (1,1)_2$ cancel the $\mathrm{U}(1)_C \cdot \mathrm{SU}(2)_L^2$ anomaly. However the $\mathrm{U}(1)_C^3$ anomaly is not canceled. We get a net $54F^3$ from $\widetilde{X}$ and $18F^3$ from the "untwisted sector" adding up to $72F^3$. Here F stands for a $\mathrm{U}(1)_C$ gauge field. The bulk 5D Chern-Simons term also contributes to the anomaly and the contribution is $-24F^3$. Altogether we get $48F^3$. The fields at the other end, $x_4 = \pi R$ contribute $-48F^3$. Therefore, locally in 5D the $\mathrm{U}(1)_C^3$ anomaly does not cancel.

But in fact the $\mathrm{U}(1)_C^3$ anomaly is not required to cancel locally in 5D. Only the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)_C$ anomalies have to cancel locally. The $\mathrm{U}(1)_C$ gauge transformation is the same with opposite signs on both ends. In other words, we are not allowed to make a different U(1) transformation at $x_4 = 0$ and at $x_4 = \pi R$ without changing the vacuum.

For the HW scenario without blowing up the fixed points, and with $R \gg 1/M_p$ we can consider the 4D world volume theory on each end of the world brane neglecting the influence of the physics at the other end. Consider the theory on the left end. This theory has a gauge symmetry with the gauge group $[E_6 \times \mathrm{SU}(3)]_L$. The untwisted sector still contributes 1/27 of the massless $[3(3, 27)]$ representation. Anomaly cancellation, thus, requires three multiplets of 3 of $\mathrm{SU}(3)_L$. Since we do not have a handle on the structure of the theory without a blow up, we can only conjecture on how such a cancellation may occur. One possibility is that the Φ field which transforms in $(\bar{3}, \bar{3})$ of the global $\mathrm{SU}_L(3) \times \mathrm{SU}_R(3)$ symmetry group in the bulk couples at the 4D left theory to the $\mathrm{SU}_L(3)$ gauge fields and the $\mathrm{SU}_R(3)$ un-gauged degrees of freedom are flavor degrees of freedom. These field cancels the $\mathrm{SU}(3)_L$ anomaly of the untwisted fields. A similar mechanism might take place on the right end of the world 4D theory. Of course, it might be that the field Φ is only a low-energy effective description of more fundamental degrees of freedom and in the regime $R \to \infty$ the anomaly cancellation mechanism is different.

7. Summary and discussion

In this paper we have addressed the strong coupling dynamics of the T^6/Z_3 (C^3/Z_3) heterotic orbifolds. The main tool used has been the duality with the strongly coupled heterotic string theory and the weakly coupled Horava Witten M-theory compactified on $(S^1/Z_2) \times (T^6/Z_3)$.

The motivation for this study has been three folded: to shed additional light on M-theory from the heterotic string, to explore the domain of large string coupling using the Horava-Witten picture and to better understand the effective action of the recently popular scenarios of 5D bulk space-time with two end-of-the-world branes. We concentrated mainly on the Z_3 orbifold that breaks the $E_8 \times E_8$ gauge group down to $\mathrm{SU}(3) \times E_6 \times \mathrm{SU}(3) \times E_6$.

We showed that the moduli space of the (C^3/Z_3) heterotic orbifold is a blow-down at the zero section of a line bundle that is the product of the two universal line bundles, $\mathcal{L}_1 \otimes \mathcal{L}_2$, over $P^2 \times P^2$. The properties of the moduli space were reproduced from the supergravity description in the large blow-up limit. The gauge instantons of the symmetric model as well as of other T^6/Z_3 orbifolds were analyzed. In the context of the E_0 theory on a segment we identified two scales of the system, namely, the compactification scale and the scale of the expectation value of the scalar field. We wrote down the $N = 1$ supersymmetric 5D E_0 theory in terms of a 4D $N = 1$ chiral and vector superfields. We then compactified this theory on (S^1/Z_2) first in the limit of an expectation value which is much larger than the inverse of the compactification scale. In this regime we reduced the 11D HW supergravity to that of a 5D theory in the form of a non-linear sigma model. We then rewrote it in terms of a linear sigma model and determined the relations between the

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linear an non-linear descriptions. Finally we were led to a conjecture about the low energy description of the five dimensional E_0-theory (the CFT that describes the the singularity region of M-theory on C^3/Z_3) compactified on S^1/Z_2.

The status of the heterotic compactifications on singular CY orbifolds stand in contrast to the situation with compactifications on singular $K3$ orbifolds. There the HW theory is weakly coupled even before the singularities are blown-up [4, 5]. Unfortunately, we still do not possess a fully coherent picture for the analogous models with compactification on singular CY orbifolds.

Another open direction is the search for possible viable phenomenological models on the 4D end-of-the-world branes. For instance one may introduce Wilson loops to get symmetry breaking patterns that are compatible with the standard model symmetries.

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A. The spectra and anomalies of the Z_3 heterotic models

We will now describe the spectra for the various orbifolds of the $E_8 \times E_8$ heterotic string theory that were discussed in the paper. The details can be found in [17]. (See also [18] for a new discussion on orbifolds.)

From the target space point of view the different orbifolds are characterized by the different embeddings of Z_3 into the gauge groups $E_8 \times E_8$. From the worldsheet point of view, the different orbifoldings are characterized by two vectors $\vec{r} = (r_1, r_2, r_3)$ and the shift vector $\vec{s} = (s_1, \ldots, s_{16})$. The conditions that these two vectors have to satisfy are

$$\sum_i r_i = \sum_i s_i = 0 \bmod 2 \qquad \sum_i r_i^2 - \sum_i s_i^2 = 0 \bmod 6 \tag{A.1}$$

In particular for $r_i = (1, 1, -2)$ the condition takes the form $\sum_i s_i^2 = 0 \bmod 6$ The nontrivial solutions of these conditions and the corresponding gauge groups are

$$\begin{aligned} (0^8; 0^8) &\rightarrow [E_8] \times [E_8] \\ (1, 1-2, 0^5; 0^8) &\rightarrow [E_8] \times [E_6 \times \mathrm{SU}(3)] \\ (1, 1-2, 0^5; 1, 1, -20^5) &\rightarrow E_6 \times \mathrm{SU}(3) \times [E_6 \times \mathrm{SU}(3)] \\ (1, 1, 0^6; -20^7) &\rightarrow [E_7 \times \mathrm{U}(1)] \times [\mathrm{SO}(14) \times \mathrm{U}(1)] \\ (1, 1, 1, 1, 2, 0^3; 20^7) &\rightarrow [\mathrm{SU}(9)] \times [\mathrm{SO}(14) \times \mathrm{U}(1)] \end{aligned} \tag{A.2}$$

The corresponding spectra of the models with broken E_8 symmetries are

$$\begin{array}{rl} U : 3(1;27,3) + 9 \text{ moduli} & T : 27[(1;27,1) + 3(1;1,\bar{3})] \\ U : 3[(27,3;1,1) + (1,1;27,3)] + 9 \text{ moduli} & T : 27(1,\bar{3};1,\bar{3}) \\ U : 3[(56;1) + 2(1;1) + (1;64) + (1;14)] + 9 \text{ moduli} & T : 27[(1;14) + 2(1;1)] \\ U : 3[(84;1) + (1;64) + (1;14)] + 9 \text{ moduli} & T : 27[(1;\bar{9})] \end{array} \tag{A.3}$$

where U adn T stand for the untwisted and twisted sectors respectively.

Anomalies of the 4D field theory were discussed in section 5 for the $E_6 \times \mathrm{SU}(3) \times [E_6 \times \mathrm{SU}(3)]$. The same anomaly cancellation mechanism applies also for the right symmetry group $[E_6 \times \mathrm{SU}(3)]$ of the second model of the above list. Since E_7 and $\mathrm{SO}(14)$ are anomaly free groups, we have to discuss only the anomalies of the $\mathrm{SU}(9)$ of the last model and of the $\mathrm{U}(1)$s of the fourth and fifth models. The contribution of the matter in the 84 representation of $\mathrm{SU}(9)$ is the same as that of 9 fundamentals. Since it is part of the untwisted sector it has to be divided by 27 so that the net contribution is that of a fundamental, since they come with a multiplicity of three. The contribution of the twisted sector is of one anti-fundamental thus one has a full anomaly cancellation.

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